What is a model of type theory?

1 Substitution calculus

We present a formal system, which at the same time can be thought of describing the syntax of basic dependent type theory, with *explicit substitutions* and a *name-free* (de Bruijn index) presentation, and defining what is a model of type theory.

We get in this way a system of combinators for wrinting terms in dependent type theory. The system of combinators we obtain is actually the same as the one used for describing cartersian closed category [4]. There has been other systems of combinators which can be used instead [3, 5], but the one we used here, given in [2] has the advantage of also being a name-free presentation of the substitution calculus formulation of type theory.

A model is given by a collection of *contexts*. If Γ, Δ are context we have a collection $\Delta \to \Gamma$ of *substitutions* from Δ to Γ. We have a substitution $1 : \Gamma \to \Gamma$ and a composition operator $\sigma\delta : \Theta \to \Gamma$ if $\delta : \Theta \to \Delta$ and $\sigma : \Delta \to \Gamma$. Furthermore we should have

$$\sigma 1 = 1\sigma = \sigma \qquad (\theta\sigma)\delta = \theta(\sigma\delta)$$

One way to express this would be that contexts form a category with substitutions as morphisms. This would be misleading however and it is better to think of this structure as an equational structure with dependent sort (more precisely, as a model of a generalized algebraic theory [1]).

If Γ is a context we have a collection of *types over* Γ. We write $\Gamma \vdash A$ to express that A is a type over Γ. If $\Gamma \vdash A$ and $\sigma : \Delta \to \Gamma$ we should have $\Delta \vdash A\sigma$. Furthermore

$$A1 = A \qquad (A\sigma)\delta = A(\sigma\delta)$$

If $\Gamma \vdash A$ we also have a collection of *elements of type* A. We write $\Gamma \vdash a : A$ to express that a is an element of type A. If $\Gamma \vdash a : A$ and $\sigma : \Delta \to \Gamma$ we should have $\Delta \vdash a\sigma : A\sigma$. Furthermore

$$a1 = a \qquad (a\sigma)\delta = a(\sigma\delta)$$

We have a *context extension operation*: if $\Gamma \vdash A$ then we have a new context $\Gamma.A$. Furthermore there is a projection $\mathsf{p} : \Gamma.A \to \Gamma$ and a special element $\Gamma.A \vdash \mathsf{q} : A\mathsf{p}$. If $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ and $\Delta \vdash a : A\sigma$ we have an extension operation $(\sigma, a) : \Delta \to \Gamma.A$. We should have

$$\mathsf{p}(\sigma, a) = \sigma \qquad \mathsf{q}(\sigma, a) = a$$

$$(\sigma, a)\delta = (\sigma\delta, a\delta) \qquad (\mathsf{p}, \mathsf{q}) = 1$$

If $\Gamma \vdash a : A$ we write $[a] = (1, a) : \Gamma \to \Gamma.A$. Thus if $\Gamma.A \vdash B$ and $\Gamma \vdash a : A$ we have $\Gamma \vdash B[a]$. If furtermore $\Gamma.A \vdash b : B$ we have $\Gamma \vdash b[a] : B[a]$. Models are usually presented by giving a class of special maps (fibrations), in our case they are the maps $\mathsf{p} : \Gamma.A \to \Gamma$, and the elements are the sections of these fibrations, in our case the maps $[a] : \Gamma \to \Gamma.A$ determined by an element $\Gamma \vdash a : A$.

2 Type system with dependent product

We suppose furthermore one operation $\Pi\ A\ B$ such that $\Gamma \vdash \Pi\ A\ B$ if $\Gamma \vdash A$ and $\Gamma.A \vdash B$. We should have $(\Pi\ A\ B)\sigma = \Pi\ (A\sigma)\ (B\sigma^+)$ where $\sigma^+ = (\sigma\mathsf{p}, \mathsf{q})$. We have an abstraction operation λb such that $\Gamma \vdash \lambda b : \Pi\ A\ B$ if $\Gamma.A \vdash b : B$. We have an application operation such that $\Gamma \vdash \mathsf{app}(c, a) : B[a]$ if $\Gamma \vdash a : A$ and $\Gamma \vdash c : \Pi\ A\ B$. These operations should satisfy the equations

$$\mathsf{app}(\lambda b, a) = b[a], \qquad c = \lambda(\mathsf{app}\ c^+), \qquad (\lambda b)\sigma = \lambda(b\sigma^+), \qquad \mathsf{app}(c, a)\sigma = \mathsf{app}(c\sigma, a\sigma)$$

where we write $c^+ = (c\mathsf{p}, \mathsf{q})$ and $\sigma^+ = (\sigma\mathsf{p}, \mathsf{q})$.

Figure 1: Rules of basic type theory

$$\frac{\Gamma \vdash}{1 : \Gamma \to \Gamma} \qquad \frac{\sigma : \Delta \to \Gamma \quad \delta : \Theta \to \Delta}{\sigma\delta : \Theta \to \Gamma}$$

$$\frac{\Gamma \vdash A \qquad \sigma : \Delta \to \Gamma}{\Delta \vdash A\sigma} \qquad \frac{\Gamma \vdash t : A \qquad \sigma : \Delta \to \Gamma}{\Delta \vdash t\sigma : A\sigma}$$

$$\frac{}{\vdash} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A}{\Gamma.A \vdash} \qquad \frac{\Gamma \vdash A}{\mathsf{p} : \Gamma.A \to \Gamma} \qquad \frac{\Gamma \vdash A}{\Gamma.A \vdash \mathsf{q} : A\mathsf{p}}$$

$$\frac{\sigma : \Delta \to \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash u : A\sigma}{(\sigma, u) : \Delta \to \Gamma.A}$$

$$\sigma 1 = \sigma \qquad 1\sigma = \sigma \qquad (\sigma\delta)\nu = \sigma(\delta\nu)$$

$$(\sigma, u)\delta = (\sigma\delta, u\delta) \qquad \mathsf{p}(\sigma, u) = \sigma \qquad \mathsf{q}(\sigma, u) = u$$

$$(\mathsf{p}, \mathsf{q}) = 1$$

3 Universe

To define a model of type theory with one universe, we assume that we have a special type $\Gamma \vdash U$ such that $U\sigma = U$ and $\Gamma \vdash A$ whenever $\Gamma \vdash A : U$. Furthermore we assume that $\Gamma \vdash \Pi\ A\ B : U$ whenever $\Gamma \vdash A : U$ and $\Gamma.A \vdash B : U$.

4 Equations

All equations we have been using can be grouped together in the equations of *C-monoid* [4]. There are the following equations of a monoid with a special constants $\mathsf{p}, \mathsf{q}, \mathsf{app}$ and operations (x, y) and λx

$$(xy)z = x(yz) \qquad x1 = 1x = x$$

$$\mathsf{p}(x, y) = x \qquad \mathsf{q}(x, y) = y \qquad (x, y)z = (xz, yz) \qquad 1 = (\mathsf{p}, \mathsf{q})$$

$$\mathsf{app}(\lambda x, y) = x[y] \qquad (\lambda x)y = \lambda(xy^+) \qquad 1 = \lambda\ \mathsf{app}$$

where we define $[y] = (1, y)$ and $x^+ = (x\mathsf{p}, \mathsf{q})$. We have $x^+(y, z) = (xy, z)$ and $x^+y^+ = (xy)^+$ and $x^+[y] = (x, y)$.

We can also describe a model of type theory with *dependent sums*. We should have $\Gamma \vdash \Sigma\ A\ B$ if $\Gamma \vdash A$ and $\Gamma.A \vdash B$. If $\sigma : \Delta \to \Gamma$ we should have $(\Sigma\ A\ B)\sigma = \Sigma\ (A\sigma)\ (B\sigma^+)$. If $\Gamma \vdash a : A$ and $\Gamma \vdash b : B[a]$ we should have $\Gamma \vdash (a, b) : \Sigma\ A\ B$. We require the equation $(a, b)\sigma = a\sigma, b\sigma$. We ask also for two operations $\Gamma \vdash \mathsf{p}c : A$ and $\Gamma \vdash \mathsf{q}c : B[\mathsf{p}c]$ if $\Gamma \vdash c : \Sigma\ A\ B$ and the equations $\mathsf{p}(a, b) = a$ and $\mathsf{q}(a, b) = b$.

5 Set-theoretic Model

Here is an example of a model. We take the collection of context to be the collection of all sets. If Γ is a set then $\Gamma \vdash A$ means that A is a family of sets indexed over the set Γ. If $\rho : \Gamma$ then $A\rho$ is a set. If $\sigma : \Delta \to \Gamma$ we define the family $\Delta \vdash A\sigma$ by the equation $(A\sigma)\rho = A(\sigma\rho)$. We can then check the equations $A1 = A$ and $(A\sigma)\delta = A(\sigma\delta)$.

We define $\Gamma \vdash a : A$ to mean that a is a section of the family A. If $\rho : \Gamma$ we have $a\rho : A\rho$. If $\sigma : \Delta \to \Gamma$ we define $a\sigma$ by the equation $(a\sigma)\rho = a(\sigma\rho)$. We can then check the equations $a1 = a$ and $(a\sigma)\delta = a(\sigma\delta)$. Indeed we have $(a1)\rho = a(1\rho) = a\rho$ and $((a\sigma)\delta)\rho = (a\sigma)(\delta\rho) = a(\sigma(\delta\rho)) = a((\sigma\delta)\rho)$.

If $\Gamma \vdash A$ we define $\Gamma.A$ to be the set of pairs ρ, u with $\rho : \Gamma$ and $u : A\rho$. We can then define p by the equation $\mathsf{p}(\rho, u) = \rho$ and q by the equation $\mathsf{q}(\rho, u) = u$.

If $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ and $\Delta \vdash a : A\sigma$ we define the extension operation $(\sigma, a) : \Delta \to \Gamma.A$ by the equation $(\sigma, a)\rho = \sigma\rho, a\rho$.

If $\Gamma \vdash A$ and $\Gamma.A \vdash B$ we define $\Gamma \vdash \Pi\ A\ B$. If $\rho : \Gamma$ then $(\Pi\ A\ B)\rho$ is the set of elements

$$w : \prod_{u:A\rho} B(\rho, u)$$

If $\Gamma.A \vdash b : B$ we define $\Gamma \vdash \lambda b : \Pi\ A\ B$ by the equation $\mathsf{app}((\lambda b)\rho, u) = b(\rho, u)$ for $\rho : \Gamma$ and $u : A\rho$. If $\Gamma \vdash a : A$ and $\Gamma \vdash c : \Pi\ A\ B$ we define $\Gamma \vdash \mathsf{app}(c, a) : B[a]$ by the equation $\mathsf{app}(c, a)\rho = \mathsf{app}(c\rho, a\rho)$ for $\rho : \Gamma$. We can then check

$$\mathsf{app}(\lambda b, a)\rho = \mathsf{app}((\lambda b)\rho, a\rho) = b(\rho, a\rho) = b[a]\rho$$

which shows that the model validates the equality $\Gamma \vdash \mathsf{app}(\lambda b, a) = b[a] : B[a]$.

6 Presheaf model

If $\mathcal{C}$ is any small category, the presheaf model of type theory over $\mathcal{C}$ can be described as follows.

We write $X, Y, Z, \dots$ the objects of $\mathcal{C}$ and $f, g, h, \dots$ the maps of $\mathcal{C}$. If $f : X \to Y$ and $g : Y \to Z$ we write gf the composition of f and g. We write $1_X : X \to X$ or simply $1 : X \to X$ the identity map of X. Thus we have $(fg)h = f(gh)$ and $1f = f1 = f$.

A context is interpreted by a presheaf Γ: for any object X of $\mathcal{C}$ we have a set $\Gamma(X)$ and if $f : Y \to X$ we have a map $\rho \longmapsto \rho f,\ \Gamma(X) \to \Gamma(Y)$. This should satisfy $\rho 1 = \rho$ and $(\rho f)g = \rho(fg)$ for $f : Y \to X$ and $g : Z \to Y$.

A type $\Gamma \vdash A$ over Γ is given by a set $A\rho$ for each $\rho : \Gamma(X)$. Furthermore if $f : Y \to X$ we have $\rho f : \Gamma(Y)$ and we can consider the set $A\rho f$. We should have a map $u \longmapsto uf,\ A\rho \to A\rho f$ which should satisfy $u1 = u$ and $(uf)g = u(fg)$.

An element $\Gamma \vdash a : A$ is interpreted by a family $a\rho : A\rho$ such that $(a\rho)f = a(\rho f)$ for any $\rho : \Gamma(X)$ and $f : Y \to X$.

This can be seen as a concrete description of what is respectively a fibration and a section of this fibration.

If $\Gamma \vdash A$ we can define a new presheaf $\Gamma.A$ by taking $(\rho, u) : (\Gamma.A)(X)$ to mean $\rho : \Gamma(X)$ and $u : A\rho$. We define $(\rho, u)f = \rho f, uf$.

If we have a map $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ we define $\Delta \vdash A\sigma$ by $(A\sigma)\rho = A\sigma\rho$.

We can interpret dependent products $\Gamma \vdash \Pi\ A\ B$ and sums $\Gamma \vdash \Sigma\ A\ B$ if we have $\Gamma \vdash A$ and $\Gamma.A \vdash B$. For $\rho : \Gamma(X)$ we define $(u, v) : (\Sigma\ A\ B)\rho$ to mean $u : A\rho$ and $v : B(\rho, u)$. We define $(u, v)f = uf, vf$ for $f : Y \to X$. On the other hand an element of $(\Pi\ A\ B)\rho$ is a family w indexed by $h : Y \to X$ with

$$wh : \prod_{u:A\rho h} B(\rho h, u)$$

and such that $\mathsf{app}(wh, u)g = \mathsf{app}(whg, ug)$ if $h : Y \to X$ and $g : Z \to Y$. We define then $(wh)f = w(hf)$. We write $w = w1$.

We can interpret $\Gamma \vdash \lambda t : \Pi\ A\ B$ whenever $\Gamma.A \vdash t : B$ and $\Gamma \vdash \mathsf{app}(v, u) : B[u]$ if $\Gamma \vdash u : A$ and $\Gamma \vdash v : \Pi\ A\ B$. Here we write $[u]$ the map $\Gamma \to \Gamma.A$ defined by $[u]\rho = \rho, u\rho$. If $\rho : \Gamma(X)$ and $f : Y \to X$ we define $\mathsf{app}((\lambda t)\rho f, a) = t(\rho f, a) : B(\rho f, a)$ for $a : A\rho f$. We take $\mathsf{app}(v, u)\rho = \mathsf{app}(v\rho, u\rho) : B(\rho, u\rho)$. We can then check that we have

$$\mathsf{app}(\lambda t, u)\rho = t(\rho, u\rho) = t[u]\rho : B(\rho, u\rho)$$

if $\Gamma.A \vdash t : B$ and $\Gamma \vdash u : A$ and $\rho : \Gamma(X)$, which shows that the model validates the conversion rule $\Gamma \vdash \mathsf{app}(\lambda t, u) = t[u] : B[u]$.

References

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