Orthogonal factorization systems in homotopy type theory*

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January 27, 2013

We show that there is a stable orthogonal factorization system $(\mathcal{E},\mathcal{M})$ where $\mathcal{E}$ is the class of n-connected functions and $\mathcal{M}$ is the class of functions with homotopy fibers of level n. Shulman[1] also worked on this topic, using arbitrary modalities instead [Shu].

1 Prerequisites

In this document we assume that Type is a univalent universe. We are typically ambiguous and universe polymorphic.

Lemma 1.1. *Suppose $f : A \to B$ and $g : B \to C$ are functions and let $c : C$. Then there is an equivalence*

$$\mathsf{hFiber}(g \circ f, c) \simeq \sum(w : \mathsf{hFiber}(g,c)),\ \mathsf{hFiber}(f, \mathsf{proj}_1\, w).$$

with underlying function $\lambda\langle a,p\rangle.\langle\langle g(a),p\rangle,\langle a,\mathsf{id}_{g(a)}\rangle\rangle$. Also, if $f,g : A \to B$ are homotopic then

$$\mathsf{hFiber}(f,b) \simeq \mathsf{hFiber}(g,b)$$

for every $b : B$. The underlying function of this equivalence is $\lambda\langle a,p\rangle.\langle a, p \bullet H(a)^{-1}\rangle$, where $H : f \sim g$ is the

Lemma 1.2. *Suppose $f : \prod(x : A),\ P(x) \to Q(x)$ is a fiberwise transformation and define $\Sigma_A f$ to be the function $\lambda\langle a,u\rangle.\langle a, f(u)\rangle : \sum(x : A),\ P(x) \to \sum(x : A),\ Q(x)$. There is an equivalence*

$$\mathsf{hFiber}(\Sigma_A f, \langle x,v\rangle) \simeq \mathsf{hFiber}(f(x), v)$$

for any $x : A$ and $v : Q(x)$.

*These are notes for a talk at the IAS during the special year on the univalent foundations of mathematics.

[1]See https://github.com/mikeshulman/HoTT/blob/master/Coq/Subcategories/ and http://golem.ph.utexas.edu/category/2011/12/reflective_subfibrations_facto.html

Definition 1.3. A function $g : A \to B$ is said to be a retract of a function $f : X \to Y$ if there is a diagram

$$\begin{CD} A @>{s}>> X @>{r}>> A \\ @V{g}VV @V{f}VV @V{g}VV \\ B @>>{s'}> Y @>>{r'}> B \end{CD}$$

for which there are

i. a homotopy $R : r \circ s \sim \mathsf{id}_A$.

ii. a homotopy $R' : r' \circ s' \sim \mathrm{idmap}_B$.

iii. a homotopy $L : f \circ s \sim s' \circ g$.

iv. a homotopy $K : g \circ r \sim r' \circ f$.

v. paths $H(a)$ witnessing the commutativity of the square

$$\begin{CD} g(r(s(a))) @>{K(s(a))}>> r'(f(s(a))) \\ @V{g(R(a))}VV @VV{r'(L(a))}V \\ g(a) @<<{R'(g(a))}< r'(s'(g(a))) \end{CD}$$

for every $a : A$. ♦

Remark 1.4. The above lemma guarantees that each $f(x)$ is of level n if and only if $\Sigma_A f$ is of level n, see the definition of homotopy levels below, and that each $f(x)$ is n-connected if and only if $\Sigma_A f$ is n-connected, see the definition of n-connectivity below. Likewise, both n-connected and n-truncated maps are closed under retracts. ⋆

Lemma 1.5. *If a function $g : A \to B$ is a retract of a function $f : X \to Y$, then* $\mathsf{hFiber}(g,b)$ *is a retract of* $\mathsf{hFiber}(f,s'(b))$ *for every $b : B$, where $s' : B \to Y$ is as in definition 1.3.*

Definition 1.6. We define the space $\mathsf{isLevel}(-2,A) \equiv \mathsf{isContr}(A)$ and

$$\mathsf{isLevel}(n+1,A) \equiv \prod(x,y:A),\ \mathsf{isLevel}(n,x \rightsquigarrow y)$$

for $n \geq -2$. A function $f : A \to B$ is said to be of homotopy level n if all of its homotopy fibers are of level n. Likewise, a dependent type is said to be of level n if all of its fibers are of level n. ♦

Proposition 1.7. *The type*

$$n\mathsf{Type} \equiv \sum(X : \mathsf{Type}),\ \mathsf{isLevel}(n, X)$$

is of homotopy level $n+1$.

For any type A there is a type $\|A\|_n$ of homotopy level n, called the n-truncation of A. This type is defined as a certain higher inductive type with one of the basic constructors being $|-|_n : A \to \|A\|_n$. We will not give this construction here and assume that the reader is already familiar with it. The useful (universal) property of $\|A\|_n$ is:

Proposition 1.8. *Suppose that A is a type and that $P : \|A\|_n \to \mathsf{Type}$ is a dependent type over $\|A\|_{n+1}$ with fibers $P(w)$ of homotopy level n. Then there is an equivalence*

$$\left(\prod(w : \|A\|_n),\ P(w)\right) \simeq \left(\prod(x : A),\ P(|x|_n)\right)$$

with underlying function $\lambda s.s \circ |-|_n$. Consequently, n-truncation has the universal property that there is an equivalence

$$(\|A\|_n \to B) \simeq (A \to B)$$

for every type B of level n, also given by precomposition with $|-|_n$.

Lemma 1.9. *For any dependent type $P : A \to \mathsf{Type}$ there is an equivalence*

$$\left\| \sum(x : A),\ P(x) \right\|_n \simeq \left\| \sum(x : A),\ \|P(x)\|_n \right\|_n$$

Proposition 1.10. *For any type A and any $n : \mathbb{N}$, there is an equivalence*

$$(|x|_{n+1} \rightsquigarrow |y|_{n+1}) \simeq \|x \rightsquigarrow y\|_n$$

for each $x, y : A$.

Proof. We begin by constructing a dependent type

$$P : \|A\|_{n+1} \to \|A\|_{n+1} \to n\mathsf{Type}.$$

Since $n\mathsf{Type}$ is of homotopy level $n+1$, it suffices to define $P(|x|_{n+1}, |y|_{n+1})$ for each $x, y : A$. So we define

$$P(|x|_{n+1}, |y|_{n+1}) \equiv \|x \rightsquigarrow y\|_n.$$

Now we have the equivalences

$$\begin{aligned}\prod(w, w' : \|A\|_{n+1}),\ P(w, w') \to (w \rightsquigarrow w') &\simeq \prod(x, y : A),\ \|x \rightsquigarrow y\|_n \to (|x|_{n+1} \rightsquigarrow |y|_{n+1}) \\ &\simeq \prod(x, y : A),\ (x \rightsquigarrow y) \to (|x|_{n+1} \rightsquigarrow |y|_{n+1}).\end{aligned}$$

We find a term of the latter type by path induction. Note that to show that $P(w,w') \simeq (w \rightsquigarrow w')$ for each $w,w' : \|A\|_{n+1}$, it suffices to show that $\sum(w' : \|A\|_{n+1}),\ P(w,w') \simeq \sum(w' : \|A\|_{n+1}),\ w \rightsquigarrow w'$. The latter space is contractible, so it suffices to show that

$$\prod(w : \|A\|_{n+1}),\ \mathsf{isContr}\big(\sum(w' : \|A\|_{n+1}),\ P(w,w')\big),$$

which is equivalent to

$$\prod(x : A),\ \mathsf{isContr}\big(\sum(w' : \|A\|_{n+1}),\ P(|x|_{n+1},w')\big)$$

The natural candidate for the center of contraction is $\langle |x|_{n+1}, |\mathsf{id}_x|_n\rangle$, i.e. we will show contractibility by showing that there is a function of type

$$\prod(w : \|A\|_{n+1})(u : P(w)),\ \langle w,u\rangle \rightsquigarrow \langle |x|_{n+1}, |\mathsf{id}_x|_n\rangle.$$

Since the type $\prod(u : P(w)),\ \langle w,u\rangle \rightsquigarrow \langle |x|_{n+1}, |\mathsf{id}_x|_n\rangle$ is of homotopy level n for each $w : \|A\|_{n+1}$, we may apply the universal property of $n+1$-truncation to reduce the problem to

$$\prod(y : A)(u : \|x \rightsquigarrow y\|_n),\ \langle |y|_{n+1}, u\rangle \rightsquigarrow \langle |x|_{n+1}, |\mathsf{id}_x|_n\rangle,$$

which is by yet another application of the universal property of n-truncation equivalent to the type

$$\prod(y : A)(u : x \rightsquigarrow y),\ \langle |y|_{n+1}, |u|_n\rangle \rightsquigarrow \langle |x|_{n+1}, |\mathsf{id}_x|_n\rangle,$$

A term of this type can be found by an application of path induction. ■

2 Orthogonal factorization in type theory

2.1 Connectivity

Definition 2.1. A function $f : A \to B$ is said to be n-connected if there is a term of type

$$\mathsf{conn}_n(f) \equiv \prod(b : B),\ \mathsf{isContr}(\|\mathsf{hFiber}(f,b)\|_n).$$

A type A is said to be n-connected if the unique function $A_! : A \to \mathsf{unit}$ is n-connected. Likewise, a dependent type $P : A \to \mathsf{Type}$ is said to be n-connected if $P(x)$ is n-connected for every $x : A$. ♦

Remark 2.2. Note that a type A is n-connected precisely when there is a term of type $\mathsf{isContr}(\|A\|_n)$. Thus, a function $f : A \to B$ is n-connected if and only if $\mathsf{hFiber}(f,b)$ is n-connected for every $b : B$. A dependent type $P : A \to \mathsf{Type}$ is n-connected precisely when $\mathsf{proj}_1 : \sum(x : A),\ P(x) \to A$ is an n-connected function.

Note that every function is -2-connected and that a function is -1-connected precisely when it is surjective.[2] forall André Joyal proposed to call n-connected functions n-covers. ⋆

[2] A function $f : A \to B$ is *surjective* if there is a term of type $\mathsf{surj}(f) := \prod(b : B),\ \|\mathsf{hFiber}(f,b)\|_1$.

Lemma 2.3. *Suppose that $f,g:A\to B$ are homotopic functions. Then f is n-connected if and only if g is n-connected.*

PROOF. Since f and g are homotopic, there is an equivalence $\mathsf{hFiber}(f,b)\simeq\mathsf{hFiber}(g,b)$ for any $b:B$ and hence $\|\mathsf{hFiber}(g,b)\|_n$ is contractible if and only if $\|\mathsf{hFiber}(f,b)\|_n$ is contractible. ∎

Lemma 2.4. *Suppose that $f:A\to B$ is n-connected and that $g:B\to C$. Then g is n-connected if and only if $g\circ f$ is n-connected.*

PROOF. Let $c:C$. Note that we have the equivalences

$$\begin{aligned}\|\mathsf{hFiber}(g\circ f,c)\|_n &\simeq \Big\|\textstyle\sum(w:\mathsf{hFiber}(g,c)),\ \mathsf{hFiber}(f,\mathsf{proj}_1\,w)\Big\|_n\\ &\simeq \Big\|\textstyle\sum(w:\mathsf{hFiber}(g,c)),\ \|\mathsf{hFiber}(f,\mathsf{proj}_1\,w)\|_n\Big\|_n\\ &\simeq \|\mathsf{hFiber}(g,c)\|_n.\end{aligned}$$

Therefore it follows that $\|\mathsf{hFiber}(g,c)\|_n$ is contractible if and only if $\|\mathsf{hFiber}(g\circ f,c)\|_n$ is contractible. ∎

A basic example of an n-connected function is the function $|-|_n:A\to\|A\|_n$ for any type A. The other prime example is the canonical function $A\to\mathsf{im}_n(f)$ for any function $f:A\to B$, where $\mathsf{im}_n(f)$ is the n-image of f. We will see the details of this example in section 2.2.

The fact that $|-|_n:A\to\|A\|_n$ is n-connected is an immediate corollary of the following proposition:

Proposition 2.5. *A function $f:A\to B$ is n-connected if and only if for every dependent type $P:B\to\mathsf{Type}$ of homotopy level n, the function*

$$\varphi:\Big(\textstyle\prod(b:B),\ P(b)\Big)\to\Big(\textstyle\prod(a:A),\ P(f(a))\Big)$$

defined by $\varphi(s)\equiv s\circ f$ is an equivalence.

PROOF. Suppose that f is n-connected and let $P:B\to\mathsf{Type}$ be a dependent type over B. Then we have the equivalences

$$\begin{aligned}\textstyle\prod(b:B),\ P(b) &\simeq \textstyle\prod(b:B),\ \|\mathsf{hFiber}(f,b)\|_n\to P(b)\\ &\simeq \textstyle\prod(b:B)(a:A)(p:f(a)\rightsquigarrow b),\ P(b)\\ &\simeq \textstyle\prod(a:A),\ P(f(a)).\end{aligned}$$

It can be checked by the reader that this equivalence is indeed given by $\lambda s.s\circ f$.

For the other direction, suppose that the function $\lambda s.s\circ f$ from $(\prod(b:B),\ P(b))$ to $(\prod(a:A),\ P(f(a)))$ is an equivalence for each dependent type $P:B\to\mathsf{Type}$ of homotopy level n. Considering the dependent type

$$b:B\vdash\|\mathsf{hFiber}(f,b)\|_n:n\mathsf{Type},$$

we obtain from the assumed equivalence a term c of type $\prod(b:B),\ \|\mathsf{hFiber}(f,b)\|_n$ with $c(f(a)) \rightsquigarrow |\langle a, \mathsf{id}_{f(a)}\rangle|_n$. To show that each $\|\mathsf{hFiber}(f,b)\|_n$ is contractible we will find a function of type

$$\prod(b:B)(w:\|\mathsf{hFiber}(f,b)\|_n),\ w \rightsquigarrow c(b).$$

Note that by the universal property of n-truncation we obtain that the above type is equivalent to

$$\prod(b:B)(a:A)(p:f(a)\rightsquigarrow b),\ |\langle a,p\rangle|_n \rightsquigarrow c(b).$$

By path induction this is equivalent to the type

$$\prod(a:A),\ |\langle a, \mathsf{id}_{f(a)}\rangle|_n \rightsquigarrow c(f(a)).$$

This property holds by our choice of $c(f(a))$. ■

Corollary 2.6. *For every number $n:\mathbb{N}$ and every type A, the canonical function $|-|_n : A \to \|A\|_n$ is n-connected.*

PROOF. For every dependent type $P : \|A\|_n \to n\mathsf{Type}$, the requested equivalence exists by proposition 1.8. ■

The following two corollaries are mere reformulations of proposition 2.5:

Corollary 2.7. *A function $f : A \to B$ is n-connected if and only if for every function $g : X \to B$ of homotopy level n, the function*

$$\varphi : \Big(\prod(b:B),\ \mathsf{hFiber}(g,b)\Big) \to \Big(\prod(a:A),\ \mathsf{hFiber}(g,f(a))\Big)$$

defined by $\varphi(s) \equiv s \circ f$ is an equivalence.

Corollary 2.8. *A function $f : A \to B$ is n-connected if and only if for every function $g : X \to B$ of homotopy level n, the function*

$$\varphi : \Big(\sum(h:B\to X),\ g\circ h \sim \mathrm{idmap}_B\Big) \to \Big(\sum(k:A\to X),\ g\circ k \sim f\Big)$$

defined by $\varphi(h,H) \equiv \langle h\circ f, H\circ f\rangle$ is an equivalence.

When we apply the above proposition to functions with codomain unit we obtain an assertion which is in itself not very interesting, but it gives some intuition on how we should look at proposition 2.5:

Corollary 2.9. *A type A is n-connected if and only if there is an equivalence*

$$(A \to B) \simeq B$$

for every type B of homotopy level n.

Lemma 2.10. *Let B be a type of homotopy level n and let $f : A \to B$ be a function. Then the induced function $g : \|A\|_n \to B$ is an equivalence if and only if f is n-connected.*

PROOF. Note that f is homotopic to $g \circ |-|_n$. By corollary 2.6 we know that $|-|_n$ is n-connected, so we get from lemma 2.4 that f is n-connected if and only if g is n-connected. Since g is a function between types of homotopy level n, the homotopy fibers of g are of level n. Hence it follows that g is n-connected if and only if g is an equivalence. ■

A useful variation to lemma 2.4 is:

Lemma 2.11. *Suppose that $f : A \to B$ is a function, that $P : A \to \mathsf{Type}$ and $Q : B \to \mathsf{Type}$ are dependent types and that $g : \prod\{a : A\},\ P(a) \to Q(f(a))$ is a fiberwise n-connected function, i.e. each $g(a)$ is assumed to be n-connected. Then the function*

$$\varphi \equiv \lambda\langle a,u\rangle.\langle f(a), g(u)\rangle : \big(\textstyle\sum(a : A),\ P(a)\big) \to \big(\textstyle\sum(b : B),\ Q(b)\big)$$

is n-connected if and only if f is n-connected.

PROOF. For $b : B$ and $v : Q(b)$ we have

$$\begin{aligned}
\|\mathsf{hFiber}(\varphi,\langle b,v\rangle)\|_n &\simeq \big\| \textstyle\sum(a : A)(u : P(a))(p : f(a) \rightsquigarrow b),\ f(p) \cdot g(u) \rightsquigarrow v \big\|_n \\
&\simeq \Big\| \textstyle\sum(w : \mathsf{hFiber}(f,b))(u : P(\mathsf{proj}_1(w))),\ g(u) \rightsquigarrow f(p)^{-1} \cdot v \Big\|_n \\
&\simeq \Big\| \textstyle\sum(w : \mathsf{hFiber}(f,b)),\ \|\mathsf{hFiber}(g(\mathsf{proj}_1 w), f(p)^{-1} \cdot v)\|_n \Big\|_n \\
&\simeq \|\mathsf{hFiber}(f,b)\|_n
\end{aligned}$$

where the transportations along $f(p)$ and $f(p)^{-1}$ are taken with respect to the dependent type Q. Therefore, both of them are contractible whenever either of them is contractible. ■

Proposition 2.12. *For every two functions $f : A \to X$ and $g : B \to X$ there is an equivalence*

$$\Big\| \textstyle\sum(x : A)(y : B),\ \|f(x) \rightsquigarrow g(y)\|_n \Big\|_{n+1} \simeq \|A\|_{n+1} \times_{\|X\|_{n+1}} \|B\|_{n+1}.$$

PROOF. We have an obvious equivalence

$$\big(\textstyle\sum(a : A)(b : B),\ \|f(a) \rightsquigarrow g(b)\|_n\big) \simeq A \times_{\|X\|_{n+1}} B$$

and the canonical map

$$A \times_{\|X\|_{n+1}} B \to \|A\|_{n+1} \times_{\|X\|_{n+1}} \|B\|_{n+1}$$

is $n+1$-connected by lemma 2.11 above (taking the functions $g(a)$ of the lemma to be the appropriate identity maps). Hence the induced function

$$\Big\| \textstyle\sum(a : A)(b : B),\ \|f(a) \rightsquigarrow g(b)\|_n \Big\|_{n+1} \to \|A\|_{n+1} \times_{\|X\|_{n+1}} \|B\|_{n+1}$$

is an equivalence. ■

Remark 2.13. It follows from the above proposition that $\|-\|_{n+1}$ cannot preserve pullbacks for $n \geq -2$, i.e. it cannot be a lex modality. As a simple example to see why -1-truncation is does not preserve pullbacks, consider the pullback of the functions $\mathsf{true}, \mathsf{false} : \mathsf{unit} \to \mathsf{bool}$. This pullback is evidently the proposition $\varnothing$, while the pullback of their -1-truncations is evidently unit. ⋆

Corollary 2.14. *For every function $f : A \to B$ and any $b : B$ there is an equivalence*

$$\Big\| \sum(a:A),\ \|f(a) \rightsquigarrow b\|_n \Big\|_{n+1} \simeq \mathsf{hFiber}(\|f\|_{n+1}, |b|_{n+1})$$

As a consequence, $\mathsf{hFiber}(\|f\|_1, y) \simeq \|A\|_1$ *for any* $y : \|B\|_1$.

Corollary 2.15. *For any dependent type $P : A \to \mathsf{Type}$ and any $n : \mathbb{N}$ there is a dependent type $Q_{n+1} : \|A\|_{n+1} \to \mathsf{Type}$ for which there are equivalences*

$$\sum(x : \|A\|_{n+1}),\ Q_{n+1}(x) \simeq \Big\| \sum(a:A),\ P(a) \Big\|_{n+1}$$

and

$$Q_{n+1}(|a|_{n+1}) \simeq \Big\| \sum(a':A)(u : Q_{n+1}(|a'|_{n+1}),\ \|a' \rightsquigarrow a\|_n \Big\|_{n+1}$$

for any $a : A$.

We leave a direct proof of this assertion to the reader. For an indirect proof we use that functions are dependent types, via fibrant replacement.

Lemma 2.16. *For any $f : A \to B$, if $\|f\|_{n+1} : \|A\|_{n+1} \to \|B\|_{n+1}$ is an equivalence, then f is n-connected.*

PROOF. Suppose we have such a function $f : A \to B$ and let $b : B$. By assumption we have that $\mathsf{hFiber}(\|f\|_{n+1}, |b|_{n+1})$ is contractible. Using lemma 2.14, we get equivalences

$$\begin{aligned} \mathsf{unit} &\simeq \Big\| \sum(a:A),\ \|f(a) \rightsquigarrow b\|_n \Big\|_{n+1} \\ &\simeq \Big\| \sum(a:A),\ \|f(a) \rightsquigarrow b\|_n \Big\|_n \\ &\simeq \|\mathsf{hFiber}(f,b)\|_n, \end{aligned}$$

showing that $\|\mathsf{hFiber}(f,b)\|_n$ is contractible for each $b : B$. ■

Lemma 2.17. *For any $f : A \to B$ and $g : B \to C$, if g and $g \circ f$ are $n+1$-connected, then $\|f\|_{n+1}$ is an equivalence and hence f is n-connected.*

PROOF. Suppose that $f : A \to B$ and $g : B \to C$ are such that g and $g \circ f$ are $n+1$-connected and let $b : B$. Then we have that $\|g\|_{n+1}$ and $\|g \circ f\|_{n+1}$ are equivalences. Since there is a homotopy $\|g \circ f\|_{n+1} \sim \|g\|_{n+1} \circ \|f\|_{n+1}$ it follows by the 3-for-2-rule that $\|f\|_{n+1}$ is an equivalence. By lemma 2.16, it follows that f is n-connected. ■

Proposition 2.18. *For every function $f : A \to B$, the codiagonal $\nabla_f : B +_A B \to B$ is $n+1$-connected whenever f is n-connected.*

PROOF. Suppose that $f : A \to B$ is n-connected. We have to show that

$$\prod(b : B),\ \mathsf{isContr}(\|\mathsf{hFiber}(\nabla_f, b)\|_{n+1})$$

For each $b : B$ we have $\langle j(b), \mathsf{id}_b\rangle : \mathsf{hFiber}(\nabla_f, b)$, so we have to show that

$$\prod(b : B)(w : \|\mathsf{hFiber}(\nabla_f, b)\|_{n+1}),\ w \rightsquigarrow |\langle j(b), \mathsf{id}_b\rangle|_{n+1}.$$

Since $\sum(b : B),\ \mathsf{hFiber}(\nabla_f, b)$ is equivalent to $B +_A B$, the above type is equivalent to the type

$$\prod(x : B +_A B),\ |\langle x, \mathsf{id}_{\nabla_f(x)}\rangle|_{n+1} \rightsquigarrow |\langle j(\nabla_f(x)), \mathsf{id}_{\nabla_f(x)}\rangle|_{n+1}.$$

We may use the induction principle of $B +_A B$. Thus, it suffices to show that

$$\begin{aligned} u &: \prod(b : B),\ |\langle i(b), \mathsf{id}_{\nabla_f(i(b))}\rangle|_{n+1} \rightsquigarrow |\langle j(\nabla_f(i(b))), \mathsf{id}_{\nabla_f(i(b))}\rangle|_{n+1} \\ v &: \prod(b : B),\ |\langle j(b), \mathsf{id}_{\nabla_f(j(b))}\rangle|_{n+1} \rightsquigarrow |\langle j(\nabla_f(j(b))), \mathsf{id}_{\nabla_f(j(b))}\rangle|_{n+1} \\ w &: \prod(a : A),\ \alpha(a) \cdot u(f(a)) \rightsquigarrow v(f(a)). \end{aligned}$$

Note that since $\nabla_f(i(b)) \equiv b$ and $\nabla_f(j(b)) \equiv b$, the above assertions simplify to

$$\begin{aligned} u &: \prod(b : B),\ |\langle i(b), \mathsf{id}_b\rangle|_{n+1} \rightsquigarrow |\langle j(b), \mathsf{id}_b\rangle|_{n+1} \\ v &: \prod(b : B),\ |\langle j(b), \mathsf{id}_b\rangle|_{n+1} \rightsquigarrow |\langle j(b), \mathsf{id}_b\rangle|_{n+1} \\ w &: \prod(a : A),\ \alpha(a) \cdot u(f(a)) \rightsquigarrow v(f(a)). \end{aligned}$$

To find u, recall that f is assumed to be n-connected. Hence it suffices to find a function of type

$$\prod(b : B),\ \|\mathsf{hFiber}(f, b)\|_n \to (|\langle i(b), \mathsf{id}_b\rangle|_{n+1} \rightsquigarrow |\langle j(b), \mathsf{id}_b\rangle|_{n+1})$$

By the universal property of n-truncation we get the equivalent type

$$\prod(b : B),\ \mathsf{hFiber}(f, b) \to (|\langle i(b), \mathsf{id}_b\rangle|_{n+1} \rightsquigarrow |\langle j(b), \mathsf{id}_b\rangle|_{n+1})$$

which is equivalent to the type

$$\prod(a : A),\ |\langle i(f(a)), \mathsf{id}_{f(a)}\rangle|_{n+1} \rightsquigarrow |\langle j(f(a)), \mathsf{id}_{f(a)}\rangle|_{n+1}$$

It suffices to find a function of type

$$\prod(a : A),\ \langle i(f(a)), \mathsf{id}_{f(a)}\rangle \rightsquigarrow \langle j(f(a)), \mathsf{id}_{f(a)}\rangle$$

We have $\alpha(a) : i(f(a)) \rightsquigarrow j(f(a))$ and $\alpha(a) \cdot \mathsf{id}_{f(a)} \rightsquigarrow \nabla_f(\alpha(a)) \rightsquigarrow \mathsf{id}_{f(a)}$ for each $a : A$, giving us the desired function u. For v we can just apply reflexivity. Because $u(f(a))$ has been defined entirely in terms of $\alpha(a)$ and canonical paths associated to it, we also get the function w automatically. ■

The other direction seems desirable, but with the current tools it is hard to achieve:

Corollary 2.19. *If a type X is n-connected then $\mathcal{S}(X)$ is $n+1$-connected.*

PROOF. A type X is n-connected if and only if the function $X_! : X \to \mathsf{unit}$ is n-connected. By the previous proposition it follows that $\nabla_{X_!} : \mathsf{unit} +_X \mathsf{unit} \to \mathsf{unit}$ is $n+1$-connected whenever X is n-connected. The type $\mathsf{unit} +_X \mathsf{unit}$ is by definition the suspension of X. ■

Corollary 2.20. *The n-sphere is $n-1$-connected.*

PROOF. By induction. The empty type $\varnothing$ is -2-connected and the 0-sphere is the suspension of $\varnothing$. The induction step follows immediately from the previous corollary because the $n+1$-sphere is the suspension of the n-sphere. ■

2.2 Orthogonal factorization through the n-image.

Definition 2.21. Suppose that $f : A \to B$ is a function. The *n-image* of f is defined as

$$\mathsf{im}_n(f) \equiv \sum(b:B),\ \|\mathsf{hFiber}(f,b)\|_n$$

We also define $\mathsf{im}_\star(f) \equiv \sum(b:B),\ \mathsf{hFiber}(f,b)$ and we denote $\mathsf{im}_1(f)$ by $\mathsf{im}(f)$. ♦

Proposition 2.22. *Let $P, Q : A \to \mathsf{Type}$ be dependent types over a type A and consider a fiberwise transformation*

$$f : \prod(a:A),\ P(a) \to Q(a)$$

from P to Q. Then $\Sigma_A f$ is n-connected if and only if each $f(a)$ is n-connected.

PROOF. Recall that we have the equivalence

$$\mathsf{hFiber}(\Sigma_A f, \langle x, v\rangle) \simeq \mathsf{hFiber}(f(x), v)$$

for each $x : A$ and $v : Q(x)$. Hence $\|\mathsf{hFiber}(\Sigma_A f, \langle x, v\rangle)\|_n$ is contractible if and only if $\|\mathsf{hFiber}(f(x), v)\|_n$ is contractible. ■

Proposition 2.23. *For any function $f : A \to B$ and any $n : \mathbb{N}$, the canonical function $\tilde{f}_n : A \to \mathsf{im}_n(f)$ is n-connected. Also, the canonical function $\tilde{f}_\star : A \to \mathsf{im}_\star(f)$ is an equivalence. Consequently, any function factors as an n-connected function followed by a function of homotopy level n.*

PROOF. Note that $A \simeq \sum(b:B),\ \mathsf{hFiber}(f,b)$. The function $\tilde{f}_n$ is the function on total spaces induced by the fiberwise transformation

$$\lambda b. |-|_n^{\mathsf{hFiber}(f,b)} : \prod(b:B),\ \mathsf{hFiber}(f,b) \to \|\mathsf{hFiber}(f,b)\|_n.$$

Since each $|-|_n$ is n-connected by corollary 2.6, the statement follows from proposition 2.22. ■

In the following proposition we set up some machinery to prove the unique factorization theorem.

Proposition 2.24. *Suppose we have a commutative diagram of functions*

$$\begin{array}{ccccc} & & X_1 & & \\ & \overset{g_1}{\nearrow} & & \overset{h_1}{\searrow} & \\ A & & & & B \\ & \underset{g_2}{\searrow} & & \underset{h_2}{\nearrow} & \\ & & X_2 & & \end{array}$$

with $H : h_1 \circ g_1 \sim h_2 \circ g_2$, where g_1 and g_2 are n-connected and where h_1 and h_2 are of homotopy level n. Then there is an equivalence

$$E(H,b) : \mathsf{hFiber}(h_1,b) \simeq \mathsf{hFiber}(h_2,b)$$

for any $b : B$ such that the underlying map $\underline{E}(H,h_1(g_1(a)))$ of $E(H,h_1(g_1(a))$ maps $\langle g_1(a), \mathsf{id}_{h_1(g_1(a))}\rangle$ to the term $\langle g_2(a), H(a)^{-1}\rangle$, for any $a : A$.

PROOF. Let $b : B$. Then we have the following equivalences:

$$\begin{aligned} \mathsf{hFiber}(h_1,b) &\simeq \sum(w : \mathsf{hFiber}(h_1,b)),\ \|\mathsf{hFiber}(g_1, \mathsf{proj}_1\, w)\|_n \\ &\simeq \Big\| \sum(w : \mathsf{hFiber}(h_1,b)),\ \mathsf{hFiber}(g_1, \mathsf{proj}_1\, w)\Big\|_n \\ &\simeq \|\mathsf{hFiber}(h_1 \circ g_1, b)\|_n \end{aligned}$$

and likewise for h_2 and g_2. Here, the first equivalence holds because g_1 is assumed to be n-connected; the second equivalence holds because h_1 is assumed to be of level n and the third equivalence holds by lemma 1.1. Also, since we have a homotopy $H : h_1 \circ g_1 \sim h_2 \circ g_2$, there is an obvious equivalence $\mathsf{hFiber}(h_1 \circ g_1, b) \simeq \mathsf{hFiber}(h_2 \circ g_2, b)$. Hence we obtain

$$\mathsf{hFiber}(h_1,b) \simeq \mathsf{hFiber}(h_2,b)$$

for any $b : B$. By analyzing the underlying functions, we get the following representation of what happens to the term $\langle g_1(a), \mathsf{id}_{h_1(g_1(a))}\rangle$ after applying each of the equivalences of which E is composed:

$$\begin{aligned} \langle g_1(a), \mathsf{id}_{h_1(g_1(a))}\rangle &\mapsto \langle\langle g_1(a), \mathsf{id}_{h_1(g_1(a))}\rangle, |\langle a, \mathsf{id}_{g_1(a)}\rangle|_n\rangle \\ &\mapsto |\langle\langle g_1(a), \mathsf{id}_{h_1(g_1(a))}\rangle, \langle a, \mathsf{id}_{g_1(a)}\rangle\rangle|_n \\ &\mapsto |\langle a, \mathsf{id}_{h_1(g_1(a))}\rangle|_n \\ &\mapsto |\langle a, H(a)^{-1}\rangle|_n \\ &\mapsto |\langle\langle g_2(a), H(a)^{-1}\rangle, \langle a, \mathsf{id}_{g_2(a)}\rangle\rangle|_n \\ &\mapsto \langle\langle g_2(a), H(a)^{-1}\rangle, |\langle a, \mathsf{id}_{g_2(a)}\rangle|_n\rangle \\ &\mapsto \langle g_2(a), H(a)^{-1}\rangle \end{aligned}$$

■

Remark 2.25. The equivalences $E(H,b)$ are such that $E(H^{-1},b) \rightsquigarrow E(H,b)^{-1}$. ⋆

Theorem 2.26. *Let $f : A \to B$ be a function. Then the space* $\mathsf{factorizations}(n, f)$ *defined by*

$$\sum(X : \mathsf{Type})(g : A \to X)(h : X \to B),\ (h \circ g \sim f) \times \mathsf{conn}_n(g) \times \mathsf{isLevel}(n, h)$$

is contractible. By proposition 2.23 we know that there is the term

$$\langle \mathsf{im}_n(f), \tilde{f}_n, \mathsf{proj}_1, \theta, \varphi, \psi \rangle : \mathsf{factorizations}(n, f)$$

where $\theta : \mathsf{proj}_1 \circ \tilde{f}_n \sim f$ is the canonical homotopy, where φ is the proof of proposition 2.23, and where ψ is the obvious proof that $\mathsf{proj}_1 : \mathsf{im}_n(f) \to B$ has homotopy fibers of level n.

PROOF. By proposition 2.23 we know that there is a term of $\mathsf{factorizations}(n, f)$, hence it is enough to show that $\mathsf{factorizations}(n, f)$ is a proposition. Suppose we have two n-factorizations

$$\langle X_1, g_1, h_1, H_1, \varphi_1, \psi_1 \rangle \qquad \text{and} \qquad \langle X_2, g_2, h_2, H_2, \varphi_2, \psi_2 \rangle$$

of f. Then we have the homotopy $H \equiv H_2^{-1} \circ H_1 : h_1 \circ g_1 \sim h_2 \circ g_2$. By the univalence axiom, it suffices to show that

i. there is an equivalence $e : X_1 \simeq X_2$,

ii. there is a homotopy $\zeta : \underline{e} \circ g_1 \sim g_2$,

iii. there is a homotopy $\eta : h_2 \circ \underline{e} \sim h_1$,

iv. there is a homotopy $H_1 \circ (h_1 \circ \zeta)^{-1} \circ (\eta \circ g_2) \sim H_2$.

where $\underline{e}$ is the function underlying the equivalence. We prove these four assertions in that order.

i. By proposition 2.24, we have a fiberwise equivalence.

$$\prod(b : B),\ \mathsf{hFiber}(h_1, b) \to \mathsf{hFiber}(h_2, b).$$

This induces an equivalence of total spaces, i.e. we have

$$\sum(b : B),\ \mathsf{hFiber}(h_1, b) \simeq \sum(b : B),\ \mathsf{hFiber}(h_2, b)$$

Of course, we also have the familiar equivalences $X_1 \simeq \sum(b : B),\ \mathsf{hFiber}(h_1, b)$ and $X_2 \simeq \sum(b : B),\ \mathsf{hFiber}(h_2, b)$. This gives us our equivalence $e(H) : X_1 \simeq X_2$. The reader may verify that the underlying function $\underline{e}(H)$ of $e(H)$ is defined by

$$\underline{e}(H, x) \equiv \mathsf{proj}_1 \underline{E}(H^{-1}, h_1(x))(\langle x, \mathsf{id}_{h_1(x)} \rangle)$$

ii. By proposition 2.24 we have $\zeta(a) : \underline{e}(H, g_1(a)) \rightsquigarrow g_2(a)$.

iii. For every $x : X_1$, we have

$$\mathsf{proj}_2\,\underline{E}(H^{-1}, h_1(x))(\langle x, \mathsf{id}_{h_1(x)}\rangle) : h_2(\underline{e}(H,x)) \rightsquigarrow h_1(x),$$

giving us a homotopy $\eta : h_2 \circ \underline{e} \sim h_1$.

iv. By proposition 2.24 we have $\eta(g_1(a)) \equiv H(a)^{-1}$ and by *ii.* we have $h_2(\zeta(a)) \equiv \mathsf{id}_{h_2(g_2(a))}$. Thus we have

$$\begin{aligned}(H_1 \circ (h_2 \circ \zeta)^{-1} \circ (\eta \circ g_1))(a) &\equiv H_1(a) \bullet h_2(\zeta(a))^{-1} \bullet \eta(g_1(a)) \\ &\equiv H_1(a) \bullet H(a)^{-1} \\ &\rightsquigarrow H_2(a).\end{aligned}$$

■

2.3 Images are stable under pullbacks

In this section we will show that pullbacks and images commute.

Lemma 2.27. *Suppose that the square*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[h]{} & D \end{array}$$

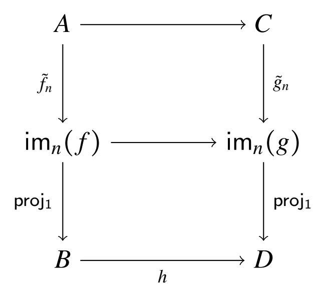
is a pullback square and let $b : B$. Then $\mathsf{hFiber}(f,b) \simeq \mathsf{hFiber}(g,h(b))$.

PROOF. This follows from pasting of pullbacks, since the type X in the diagram

$$\begin{array}{ccccc} X & \longrightarrow & A & \longrightarrow & C \\ \downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathsf{unit} & \xrightarrow[b]{} & B & \xrightarrow[h]{} & D \end{array}$$

is the pullback of the left square if and only if it is the pullback of the outer rectangle: $\mathsf{hFiber}(f,b)$ is the pullback of the square on the left and $\mathsf{hFiber}(g,h(b))$ is the pullback of the outer rectangle. ■

Theorem 2.28. *Consider functions $f : A \to B$, $g : C \to D$ and the diagram*



Then the outer rectangle is a pullback if and only if the bottom square is a pullback. In either of these equivalent cases, the top square is also a pullback. Consequently, images are stable under pullbacks.

Proof. Suppose first that the outer square is a pullback. Note that we have the equivalences

$$\begin{aligned} B \times_D \mathsf{im}_n(g) &\equiv \sum (b:B)(w:\mathsf{im}_n(g)),\ h(b) \rightsquigarrow \mathsf{proj}_1\, w \\ &\simeq \sum (b:B)(d:D)(w:\|\mathsf{hFiber}(g,d)\|_n),\ h(b) \rightsquigarrow d \\ &\simeq \sum (b:B),\ \|\mathsf{hFiber}(g,h(b))\|_n. \\ &\simeq \sum (b:B),\ \|\mathsf{hFiber}(f,b)\|_n \\ &\equiv \mathsf{im}_n(f). \end{aligned}$$

In the last equivalence we have used lemma 2.27.

Now suppose that the bottom square is a pullback, of which we denote the top arrow by ψ. By the pasting lemma for pullbacks, it suffices to show that the top square is a pullback. We have the equivalences

$$\begin{aligned} \mathsf{im}_n(f) \times_{\mathsf{im}_n(g)} C &\equiv \sum (w:\mathsf{im}_n(f))(c:C),\ \psi(w) \rightsquigarrow \tilde{g}_n(c) \\ &\simeq \sum (b:B)(w:\mathsf{im}_n(g))(p:h(b) \rightsquigarrow \mathsf{proj}_1\, w)(c:C),\ w \rightsquigarrow \tilde{g}_n(c) \\ &\simeq \sum (b:B)(c:C),\ h(b) \rightsquigarrow g(c) \\ &\simeq \sum (b:B),\ \mathsf{hFiber}(f,b) \\ &\simeq A. \end{aligned}$$

■

Corollary 2.29. *The class of n-surjective functions is stable under pullbacks.*

By analogy to what happens in an ∞-topos [REZ10, Lur09], Shulman defines internally[3] a reflective subcategory R of the category Type such that for all X in R and $P : X \to R$, $\Sigma X P$ in R. He shows that these reflections are equivalent to modalities.

[3]We should be careful here, as we do not yet properly understand what should be the ∞-category of Type.

Any such modality $\bigcirc$ should define a stable factorization system with $\mathcal{E}$ the class of functions such that:

$$\mathsf{E}_{\bigcirc}(f) \equiv \prod(b:B),\ \mathsf{isContr}(\bigcirc \mathsf{hFiber}(f,b)).$$

and $\mathsf{M}_{\bigcirc}$ the class of functions all whose fibers are modal. Our truncation is a special case of this where we consider the modality $\| - \|_n$. Shulman defines the type of these maps and the lift (in Coq). We plan to check whether our proof of Theorem 2.26 generalizes to modalities.

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