Notes on Models of Type Theory

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(1) Def. (Dybjer via Coquand)

CwF

A category w/ families consists of:

(i) a category $\mathbb{C}$ (of "contexts"), written

$$\sigma : \Delta \to \Gamma$$

(ii) for each Γ, a collection of "types"

$$\mathrm{Types}(\Gamma) \ni A$$

written $\quad \Gamma \vdash A$,

with action by $\sigma : \Delta \to \Gamma$

denoted

$$\Delta \vdash A\sigma$$

s.th.

$$A1 = A$$

$$(A\sigma)\tau = A(\sigma\tau) \, .$$

(iii) • for each $\Gamma \vdash A$, a collection of "elements"

$$\mathrm{Elem}(\Gamma, A) \ni a$$

written

$$\Gamma \vdash a : A \quad ,$$

with action by $\sigma : \Delta \to \Gamma$ denoted

$$\Delta \vdash a\sigma : A\sigma$$

s.th.

$$a1 = a$$
$$a(\sigma\tau) = (a\sigma)\tau \quad .$$

(iv) • for each $\Gamma \vdash A$, an "extension"

/ $\Gamma . A \quad \in \quad \mathrm{ob}\,\mathcal{C}$,

/ $p : \Gamma . A \to \Gamma \quad \in \quad \mathrm{ar}\,\mathcal{C}$,

/ q where $\Gamma . A \vdash q : Ap$.

• for each $\sigma : \Delta \to \Gamma$, $\Delta \vdash a : A\sigma$, a map:

$$(\sigma, a) : \Delta \longrightarrow \Gamma . A \quad .$$

These should satisfy :

$$p \circ (\sigma, a) = \sigma$$

$$q(\sigma, a) = a$$

$$(\sigma, a) \circ \delta = (\sigma \circ \delta, a\delta)$$

$$(p, q) = 1 .$$

(2) These data specify :

Presheaves

(i) a cat $\mathcal{C}$

(ii) a presheaf $T : \mathcal{C}^{op} \to Set$

(iii) a presheaf $E : \mathcal{C}^{op} \to Set$
together with a nat. map

$$p : E \to T .$$

Let's see how this works before doing (iv).

Recall the Yoneda Lemma : for any P & C :

$$\frac{x \in PC}{\overline{x : yC \to P}}$$

So we can write

$$\Gamma \vdash A$$

as

$$y\Gamma \xrightarrow[A]{} T \qquad \text{in } \hat{\mathcal{C}}$$

Moreover, given

$$\begin{tikzcd} E(T) \arrow[d, "p_T"] \\ A \in T(\Gamma) \end{tikzcd}$$

let $E(\Gamma, A) = p_T^{-1}(A)$,

then we have

$$\Gamma \vdash a : A$$

$\Longleftrightarrow$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ y\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & T \end{tikzcd}$$

<u>Def</u> (Grothendieck) A map of presheaves

$$f : Y \to X$$

is <u>representable</u> if all of its fibers are representable, in the sense:

for all $A \in \mathcal{C}$ & $x \in X(A)$, there's $B \xrightarrow{f_x} A$ in $\mathcal{C}$

s.th.:

$$\begin{tikzcd}
yB \arrow[r] \arrow[d, "y f_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
yA \arrow[r, "x"'] & X
\end{tikzcd}$$

<u>Prop.</u> If $\mathcal{C}$, $p : E \to T$ is a CwF,

then p is representable.

Pf. Take

$$\begin{tikzcd}
y\Gamma.A \arrow[r, "q"] \arrow[d, "y p_A"'] & E \arrow[d, "p"] \\
y\Gamma \arrow[r, "A"'] & T
\end{tikzcd}$$

.

Thm The converse is also true:

if $\mathcal{C}$ & $p : E \to T$ is representable,

then there are operations $(-,-)$, q

making $\mathcal{C}, E, T, p, (-,-), q$ a CwF.

pf. Use

$$\begin{tikzcd}
\int_{\mathcal{C}} T \arrow[rr, "p"] \arrow[dr, "\pi"'] & & \mathcal{C}^{\to} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

.

(3) **Length** The equivalence between the notions of "CwF" & "representable map of presheaves" makes it clear how many different such things there are.

Note that composites of representable maps are representable, & these are also closed under pullback, coproducts, and ...

One way to restrict such structures is by requiring a "length operation".

Def A __length__ function on a CwF is

$$l : \mathcal{C}_0 \longrightarrow \mathbb{N}$$

s.th.

- $$l(\Gamma) = 1 \iff \Gamma = 1$$

  (term. obj.)

- for $\Gamma \vdash A$,

  $$l(\Gamma . A) = l(\Gamma) + 1$$

- for any $l(\Delta) \neq 0$, there's a unique Γ & $\Gamma \vdash A$ s.th.

  $$\Delta = \Gamma . A$$

  $$\left(\text{\& so,} \quad l\Delta = l\Gamma.A = l\Gamma + 1 \right).$$

So in particular, every object Δ of $\mathcal{C}$ is generated from 1 by context extension $\Gamma . A$.

Now let $p: E \to T$ be representable, and form:

$$
\begin{tikzcd}
 & & \mathcal{C} & \\
(T', A) & \mathbb{T} = \int_{\mathcal{C}} T \arrow[r, "p"] \arrow[ur, bend left, "p_2"] \arrow[dr, bend right, "p_1"'] & \mathcal{C}^{\to} \arrow[u, "\mathrm{dom}"'] \arrow[d, "\mathrm{cod}"] & T'.A \arrow[d, "p_A"] \\
 & & \mathcal{C} & T'
\end{tikzcd}
$$

<u>Def.</u> A length (<u>grading</u>) on $(\mathcal{C}, p)$ is

$$\ell : \mathcal{C}_0 \longrightarrow \mathbb{N}$$

s.th.

(1)

$$
\begin{tikzcd}
\mathbb{T}_0 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_0 \arrow[r, "\ell"] & \mathbb{N} \arrow[d, "s"] \\
\mathcal{C}_0 \arrow[rr, "\ell"'] & & \mathbb{N}
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "0"] \\
\mathcal{C}_0 \arrow[r, "\ell"'] & \mathbb{N}
\end{tikzcd}
$$

<u>Prop.</u> These notions of "length" are equivalent.

Note that given a grading on

$$p : E \longrightarrow T$$

we have a p.b. :

$$\begin{tikzcd}
\Pi_0 + 1 \arrow[r, "\{l p_2 + 1\}"] \arrow[d, "(P_1, 1)"', "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N} + 1 \arrow[d, "\wr", "(S, 0)"'] \\
\mathcal{C}_0 \arrow[r] & \mathbb{N}
\end{tikzcd}$$

Therefore :

$$\mathcal{C}_0 \cong \Pi_0 + 1$$