

Notes on Models of Type Theory

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(1) Def. (Dybjer via Coquand)

CwF

A category w/ families consists of:

(i) a category \mathcal{C} (of "contexts"), with

$$\sigma: \Delta \rightarrow \Gamma$$

(ii) for each Γ , a collection of "types"

$$\text{Types}(\Gamma) \ni A$$

with

$$\Gamma \vdash A$$

with actn by $\sigma: \Delta \rightarrow \Gamma$
denoted

$$\Delta \vdash A\sigma$$

S.t.

$$A \perp = A$$

$$(A\sigma)\tau = A(\sigma\tau)$$

(iii). for each $\mathbb{T} \vdash A$, a collection of elements

$$\text{Elem}(\mathbb{T}, A) \ni a$$

written

$$\mathbb{T} \vdash a : A$$

with action by $\sigma : \Delta \rightarrow \mathbb{T}$
denoted

$$\Delta \vdash a\sigma : A\sigma$$

s.t.

$$a1 = a$$

$$a(\sigma\tau) = (a\sigma)\tau$$

(iv). for each $\mathbb{T} \vdash A$, an extension

$$/ \quad \mathbb{T}.A \quad \text{in } \text{ob } \mathcal{C},$$

$$/ \quad p : \mathbb{T}.A \rightarrow \mathbb{T} \quad \text{in } \text{ar } \mathcal{C},$$

$$/ \quad q \quad \text{where} \quad \mathbb{T}.A \vdash q : A_p.$$

• for each $\sigma : \Delta \rightarrow \mathbb{T}$, $\Delta \vdash a : A\sigma$,
a map:

$$(\sigma, a) : \Delta \rightarrow \mathbb{T}.A$$

These should satisfy:

$$p \circ (\sigma, a) = \sigma$$

$$q(\sigma, a) = a$$

$$(\sigma, a) \circ \delta = (\sigma \circ \delta, a \delta)$$

$$(p, q) = 1$$

(2) These data specify:

Presheaves

(i) a cat \mathcal{C}

(ii) a presheaf $T: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

(iii) a presheaf $E: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$
together with a nat. map

$$p: E \rightarrow T$$

Let's see how this works before doing (iv).

Recall the Yoneda Lemma: for any $P \& C$:

$$\underline{\underline{x \in PC}}$$

$$x: y \in C \longrightarrow P$$

So we can write

$$\pi + A$$

as

$$y\pi \xrightarrow[A]{} T \quad \text{in } \hat{\mathcal{G}}$$

Moreover, given $E(\pi)$

$$\downarrow P_\pi$$

$$A \in T(\pi)$$

Let $E(\pi, A) = P_\pi^{-1}(A)$,

then we have

$$\pi + a : A$$

\Leftrightarrow

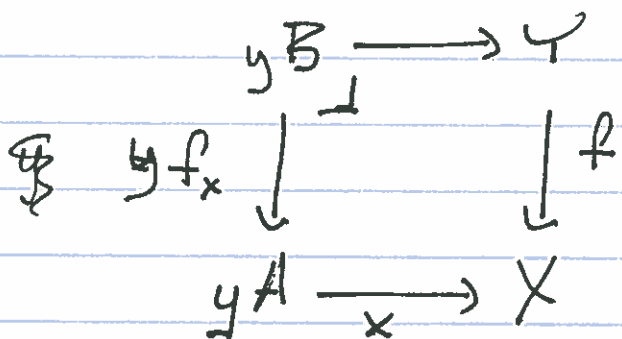
$$\begin{array}{ccc} & & E \\ & \nearrow a & \downarrow P \\ y\pi & \xrightarrow[A]{} & T \end{array}$$

Def (Grothendieck) A map of presheaves

$$f : Y \rightarrow X$$

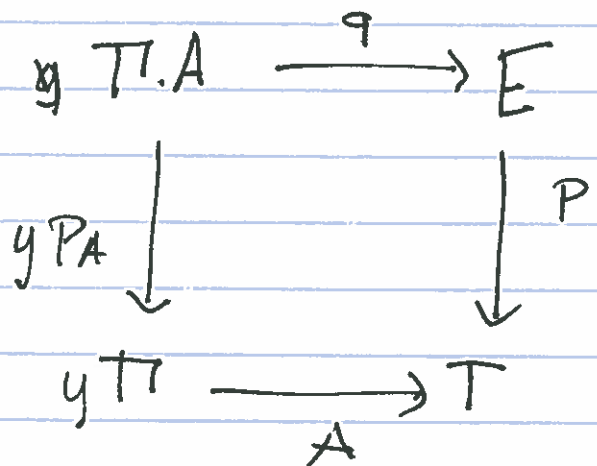
is representable if all of its fibers are representable, in the sense:

for all $A \in \mathcal{C}$ & $x \in X(A)$, there's $f_x \downarrow$
s.t.l. $\in \mathcal{C}$ A



Prop. If \mathcal{C} , $p : E \rightarrow T$ is a CwF, then p is representable.

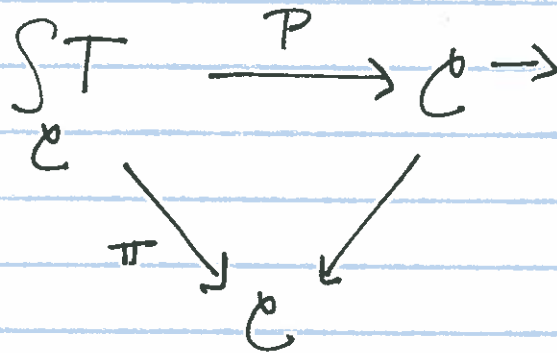
Pf. Take



Then the converse is also true :

if \mathcal{C} & $p: E \rightarrow T$ is representable, then there are operations $(-, -), \eta$ making $\mathcal{C}, E, T, p, (-, -), \eta$ a CwF.

pf. Use



(3)
Length

the equivalence between the notions of "CwF" & "representable map of presheaves" makes it clear how many different such things there are. Note that composites of representable maps are representable, & these are also closed under pullback, coproducts, and ...

One way to restrict such structures is by requiring a "length operation".

Def A length function on a CwF is

$$l: \mathcal{C}_0 \longrightarrow \mathbb{N}$$

s.t.h.

$$\bullet \quad l(\mathbb{1}) = 1 \iff \mathbb{1} = \mathbb{1} \\ \text{(term. obj.)}$$

• for $\Gamma \vdash A$,

$$l(\Gamma.A) = l(\Gamma) + 1$$

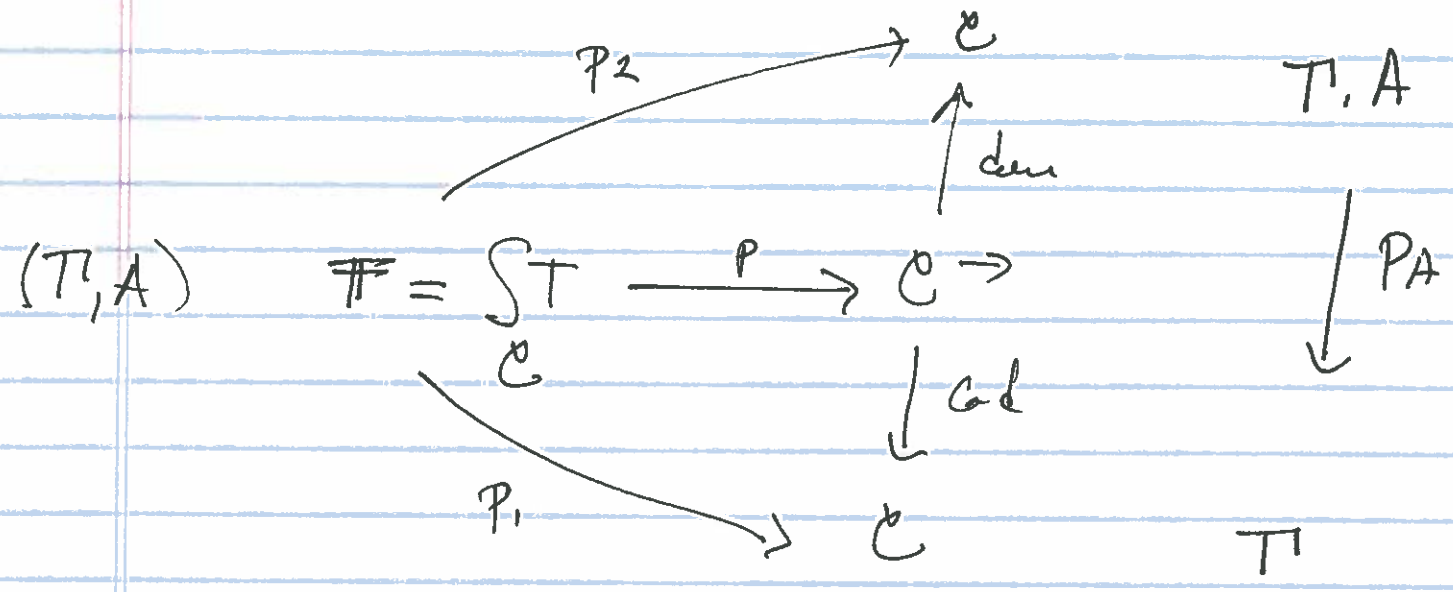
• for any $l(\Delta) \neq 0$, there's a unique $\Gamma \nVdash \Gamma \vdash A$ s.t.h.

$$\Delta = \Gamma.A$$

$$\left(\& \text{s.t. } l \Delta = l \Gamma.A = l \Gamma + 1 \right).$$

So in particular, every object Δ of \mathcal{C} is generated from $\mathbb{1}$ by context extension $\Gamma.A$.

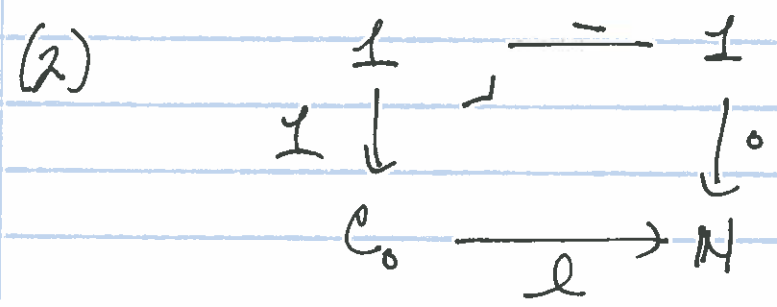
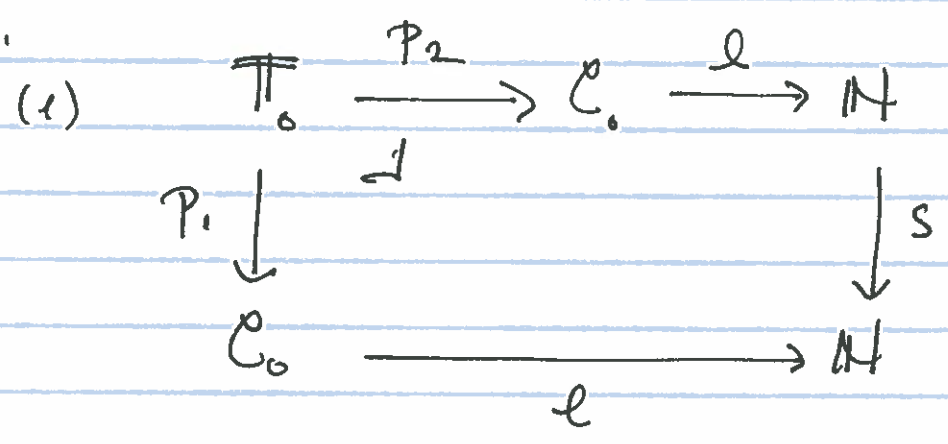
Now let $p: E \rightarrow T$ be representable, and form:



Def. A length (grading) on (\mathbb{C}, p) is

$$l: \mathbb{C}_0 \rightarrow \mathbb{N}$$

S.th.



Prop. These notions of "length" are equivalent.

Note that given a grading on $\mathcal{P}: E \rightarrow T$

we have a p.b.:

$$\begin{array}{ccc}
 \mathbb{T}_0 + 1 & \xrightarrow{\mathbb{Z} \langle p_{2+1} \rangle} & \mathbb{N} + 1 \\
 (p, 1) \downarrow \mathbb{Z} & \lrcorner & \downarrow \mathbb{Z} (s, 0) \\
 \mathcal{C}_0 & \xrightarrow{\quad\quad\quad} & \mathbb{N}
 \end{array}$$

Therefore:

$$\mathcal{C}_0 \cong \mathbb{T}_0 + 1$$