The fundamental group of the circle is isomorphic to $\mathbb{Z}$

Working group on informal type theory

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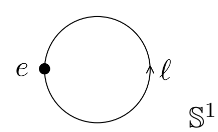
The aim of this paper is to prove that the fundamental group of the circle is isomorphic to the group of integers $\mathbb{Z}$. Recall that the circle $\mathbb{S}^1$ is the space generated by a point e and a path ℓ from e to itself:

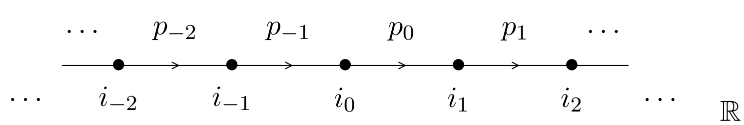


We will actually prove a bit more, namely that the loop space of $\mathbb{S}^1$ at e is equivalent to $\mathbb{Z}$. Using the fact that $\mathbb{Z}$ is a set, the result will follow easily.

We start by constructing a particular fibration over $\mathbb{S}^1$:

Definition 1. *Let $\widetilde{\mathbb{S}^1} \twoheadrightarrow \mathbb{S}^1$ be the fibration over $\mathbb{S}^1$ whose fiber over e is $\mathbb{Z}$ and such that for every $n \in (\widetilde{\mathbb{S}^1})_e = \mathbb{Z}$, transporting n along ℓ gives $n+1$. Such a fibration exists because the map $(n \mapsto n+1) : \mathbb{Z} \to \mathbb{Z}$ is an equivalence.*

The total space of this fibration is not very explicit and difficult to work with so we will first show that it has a more explicit presentation. Let $\mathbb{R}$ be the space generated by a point i_n for every $n \in \mathbb{Z}$ and a path p_n from i_n to i_{n+1} for every $n \in \mathbb{Z}$:



Then we have the following:

Lemma 2 (flattening lemma)**.** *The spaces $\widetilde{\mathbb{S}^1}$ and $\mathbb{R}$ are equivalent.*

The idea is that the total space of the fibration $\widetilde{\mathbb{S}^1} \twoheadrightarrow \mathbb{S}^1$ can be *flattened* to give the space $\mathbb{R}$ (whose definition can be deduced from the definition of $\widetilde{\mathbb{S}^1} \twoheadrightarrow \mathbb{S}^1$). A similar lemma can be proved every time we have a space A generated by some points and paths constructors and a fibration $Q \twoheadrightarrow A$ defined by giving the fibers at the distinguished points and the action of the distinguished paths on the fibers.

Proof. We will construct two functions $f : \mathbb{R} \to \widetilde{\mathbb{S}^1}$ and $g : \widetilde{\mathbb{S}^1} \to \mathbb{R}$ and we will prove that they are inverse to each other.

For f we take the function sending each point i_n to $\widehat{n}$, where $n \mapsto \widehat{n}$ is the inclusion $\mathbb{Z} = (\widetilde{\mathbb{S}^1})_e \hookrightarrow \widetilde{\mathbb{S}^1}$, and sending each path p_n to the path in $\widetilde{\mathbb{S}^1}$ above ℓ corresponding to the transport of n to $n+1$.

To define g, we will define $g_x : (\widetilde{\mathbb{S}^1})_x \to \mathbb{R}$ for every $x \in \mathbb{S}^1$. For $x = e$, we have $(\widetilde{\mathbb{S}^1})_e = \mathbb{Z}$ and we define $g_e : \mathbb{Z} \to \mathbb{R}$ by $g_e(n) = i_n$. When x varies along l, the map g_x varies from g_e to the map $g'_e : \mathbb{Z} \to \mathbb{R}$ such that $g'_e(n) = i_{n+1}$ (by definition of the fibration $\widetilde{\mathbb{S}^1} \twoheadrightarrow \mathbb{S}^1$), so we need a path from i_n to i_{n+1} for every $n \in \mathbb{Z}$ and we take p_n.

We now prove that f and g are inverse to each other.

Let's first consider the composite $g \circ f : \mathbb{R} \to \mathbb{R}$. For $n \in \mathbb{Z}$, we have $g(f(i_n)) = g(\widehat{n}) = i_n$. We also have $g(f(p_n)) = p_n$ because by definition g sends the path in $\widetilde{\mathbb{S}^1}$ above ℓ corresponding to the transport of n to $n+1$ to p_n. Hence $g \circ f = \mathrm{id}_{\mathbb{R}}$.

For the other composite, we will prove that for all $x \in \mathbb{S}^1$, the map $f \circ g_x : (\widetilde{\mathbb{S}^1})_x \to \widetilde{\mathbb{S}^1}$ is equal to the inclusion $(\widetilde{\mathbb{S}^1})_x \hookrightarrow \widetilde{\mathbb{S}^1}$. For $x = e$, we have $(\widetilde{\mathbb{S}^1})_e = \mathbb{Z}$ and $f(g_e(n)) = \widehat{n}$ for all $n \in \mathbb{Z}$. When x varies along ℓ, the inclusion $(\widetilde{\mathbb{S}^1})_x \hookrightarrow \widetilde{\mathbb{S}^1}$ varies from $n \mapsto \widehat{n}$ to $n \mapsto \widehat{n+1}$. Moreover, $f \circ g_x$ varies from $f \circ g_e$ to the map $f \circ g'_e$ and for $n \in \mathbb{Z}$ we have $f(g'_e(n)) = f(i_{n+1}) = \widehat{n+1}$. □

Lemma 3. *The space $\widetilde{\mathbb{S}^1}$ is contractible.*

Proof. Thanks to the flattening lemma, we can just prove that $\mathbb{R}$ is contractible. Hence we want to construct for every $x \in \mathbb{R}$ a path c_x from i_0 to x.

For $x = i_0$, we take the constant path:

$$c_{i_0} = 1_{i_0}$$

For $x = i_{n+1}$ with $n \geq 0$, we take the composite of the path from i_0 to i_n (which has already been constructed by induction on n) with p_n:

$$c_{i_{n+1}} = c_{i_n} \star p_n \qquad (n \geq 0)$$

For $x = i_n$ with $n < 0$, we take the composite of the path from i_0 to i_{n+1} with the opposite of p_n:

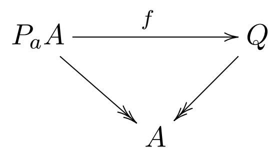
$$c_{i_n} = c_{i_{n+1}} \star \overline{p_n} \qquad (n < 0)$$

Finally we need to prove that this choice of paths is stable by composition with the paths p_n: we need to prove that $c_{i_n} \star p_n = c_{i_{n+1}}$ for all $n \in \mathbb{Z}$. This is true by definition for $n \geq 0$ and for $n < 0$ we have $c_{i_n} \star p_n = c_{i_{n+1}} \star \overline{p_n} \star p_n = c_{i_{n+1}}$. □

The previous property is very interesting because of the following lemma:

Lemma 4. *Let (A, a) be a pointed space and $Q \twoheadrightarrow A$ a fibration over A with Q_a pointed. If Q is contractible, then $\Omega_a A$ (the space of paths from a to itself in A) is equivalent to Q_a.*

Proof. Consider the following commutative diagram



where $P_a A \twoheadrightarrow A$ is the fibration whose fiber over $x \in A$ is the space of paths from a to x and f is the map transporting along its argument the distinguished point of Q_a. The total space of the fibration $P_a A \twoheadrightarrow A$ is the space of pairs (x, p) where $x \in A$ and p is a path from a to x. This space is contractible because every such pair is equal to the pair $(a, 1_a)$ by moving back along $\overline{p}$.

The map f is then a map between contractible spaces, so it's an equivalence. Given that the diagram is commutative, $P_a A \to A$ and $Q \to A$ are equivalent when considered as maps with codomain A, so they have the same homotopy fiber at a. But these maps are fibrations, hence homotopy fibers are just fibers and $(P_a A)_a \simeq \Omega_a A \simeq Q_a$. □

We can now deduce the fundamental group of the circle.

Theorem 5. *We have $\Omega_e \mathbb{S}^1 \simeq \mathbb{Z}$ and $\pi_1(\mathbb{S}^1, e) \simeq \mathbb{Z}$ (as spaces).*

Proof. The first result is obtained by applying the previous lemma to the pointed space $(\mathbb{S}^1, e)$ and the fibration $\widetilde{\mathbb{S}^1} \twoheadrightarrow \mathbb{S}^1$ (where $(\widetilde{\mathbb{S}^1})_e = \mathbb{Z}$ is pointed by 0) and using lemma 3 and the fact that $(\widetilde{\mathbb{S}^1})_e = \mathbb{Z}$.

For the second result, recall that $\pi_1(A, a)$ is defined as the set of connected components of $\Omega_a A$. But $\mathbb{Z}$ is a set, hence we also have $\pi_1(\mathbb{S}^1, e) \simeq \mathbb{Z}$. □

Theorem 6. *We have $\pi_1(\mathbb{S}^1, e) \simeq \mathbb{Z}$ (as groups).*

Proof. By definition, the map $\pi_1(\mathbb{S}^1, e) \to \mathbb{Z}$ applied to some loop p corresponds to transporting $\widehat{0}$ along p in the fibration $\widetilde{\mathbb{S}^1} \twoheadrightarrow \mathbb{S}^1$. In particular, if $p = \ell^{\star n}$ is an iterated composition of the distinguished loop ℓ, for $n \in \mathbb{Z}$, the image of p in $\mathbb{Z}$ is n.

Hence every element of $\pi_1(\mathbb{S}^1, e)$ is equal to $\ell^{\star n}$ for some $n \in \mathbb{Z}$, and $\ell^{\star n} \star \ell^{\star m} = \ell^{\star(n+m)}$ by an easy induction on n, so the map $\pi_1(\mathbb{S}^1, e) \to \mathbb{Z}$ is a group isomorphism. □